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A MULTIMODULUS ELASTICITY THEORY

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This paper considers certain questions regarding the application of elasticity theory to bodies whose materials resist extension and compression in different ways. Some formulas and theorems of the classical theory of elasticity are presented and their validity is demonstrated for the considered material. /64*

1. The multimodulus theory of elasticity for isotropic elastic materials with the elastic characteristics E^+ and ν^+ (upon extension in any direction) and E^- and ν^- (upon compression in any direction), as is known [1, 2], differs from the classical elasticity theory only by the law of elasticity applied. In the multimodulus theory of elasticity, the generalized law of elasticity in a rectangular system of coordinates x, y, z has the following form [1, 2]:

$$\begin{array}{ll} e_{xx} = a_{11}\sigma_x + a_{12}(\sigma_y + \sigma_z) + B_3 m_1^2 \sigma_\beta & e_{xy} = 2A_1\tau_{xy} + 2B_3 m_1 m_2 \sigma_\beta \\ e_{yy} = a_{11}\sigma_y + a_{12}(\sigma_x + \sigma_z) + B_3 m_2^2 \sigma_\beta & e_{xz} = 2A_1\tau_{xz} + 2B_3 m_1 m_3 \sigma_\beta \\ e_{zz} = a_{11}\sigma_z + a_{12}(\sigma_x + \sigma_y) + B_3 m_3^2 \sigma_\beta & e_{yz} = 2A_1\tau_{yz} + 2B_3 m_2 m_3 \sigma_\beta \end{array} \qquad (1.1)$$

where, along with the known designations [3], we also have

$$A_1 = a_{11} - a_{12}, \qquad B_3 = a_{22} - a_{11} \qquad (1.2)$$

$$m_1 = \cos(x, \beta), \qquad m_2 = \cos(y, \beta), \qquad m_3 = \cos(z, \beta) \qquad (1.3)$$

Here β is one of the principal directions of the stresses and strains at a given point; σ_β is the principal stress which corresponds to the principal direction β; m_i are the direction cosines.

In order to make the following discussion more specific, the generalized law of elasticity (1.1) is written for a case when the principal direction σ_β has a sign which is different from that of the other two principal stresses σ_α and σ_γ, i.e., when $\sigma_\beta < 0$, $\sigma_\alpha > 0$, and $\sigma_\gamma > 0$, due to which we have $a_{11} = 1/E^+$, $a_{22} = 1/E^-$, $a_{12} = -\nu^+/E^+ = -\nu^-/E^-$ for the elasticity factors a_{ik}; or when $\sigma_\beta > 0$, $\sigma_\gamma < 0$, $\sigma_\alpha < 0$, due to which we have $a_{11} = 1/E^-$, $a_{22} = 1/E^+$, and $a_{12} = -\nu^+/E^+ = -\nu^-/E^-$ for a_{ik}.

*Numbers in the margin indicate pagination in the foreign text.

Solving system (1.1) with respect to the stresses, we obtain

$$\begin{aligned} \sigma_x &= 2Ae_{xx} + B\theta - B_3(B + 2Am_1^2)\sigma_\beta, & \tau_{xy} &= A(e_{xy} - 2B_3m_1m_2\sigma_\beta) \\ \sigma_y &= 2Ae_{yy} + B\theta - B_3(B + 2Am_2^2)\sigma_\beta, & \tau_{xz} &= A(e_{xz} - 2B_3m_1m_3\sigma_\beta) \\ \sigma_z &= 2Ae_{zz} + B\theta - B_3(B + 2Am_3^2)\sigma_\beta, & \tau_{yz} &= A(e_{yz} - 2B_3m_2m_3\sigma_\beta) \end{aligned} \tag{1.4}$$

$$A = \frac{1}{2A_1}, \qquad B = -\frac{a_{12}}{A_1(a_{11} + 2a_{12})}, \qquad \theta = e_{xx} + e_{yy} + e_{zz} \tag{1.5}$$

We obtain the following for σ_β in strains by direct calculation [1]:

$$\sigma_\beta = \frac{2Ae_{\beta\beta} + B\theta}{1 + B_3(2A + B)} \tag{1.6}$$

The generalized law of elasticity (1.1) is easily expressed in the principal directions of stresses and strains α, β, γ:

$$\begin{aligned} e_{\alpha\alpha} &= a_{11}\sigma_\alpha + a_{12}(\sigma_\beta + \sigma_\gamma) \\ e_{\beta\beta} &= a_{22}\sigma_\beta + a_{12}(\sigma_\alpha + \sigma_\gamma) \\ e_{\gamma\gamma} &= a_{11}\sigma_\gamma + a_{12}(\sigma_\alpha + \sigma_\beta) \end{aligned} \tag{1.7}$$

The relative position of the principal directions α, β, γ with respect to the coordinate axes x, y, z at a given point are determined by means of nine direction cosines (see diagram) which are related by the following dependences*: /65

$$\begin{array}{cccc} & \alpha & \beta & \gamma \\ x & l_1 & m_1 & n_1 \\ y & l_2 & m_2 & n_2 \\ z & l_3 & m_3 & n_3 \end{array} \qquad \begin{aligned} & l_i^2 + m_i^2 + n_i^2 = 1 \quad (i = 1, 2, 3) \\ & l_1m_1 + l_2m_2 + l_3m_3 = 0 \quad (l\ m\ n) \end{aligned} \tag{1.8}$$

We shall present known formulas for the transformation of the stress and strain components at a given point in the transition from one coordinate system to another, related by diagram (1.8), under the condition that α, β, and γ are the principal directions of the stresses and strains:

for the strain components

$$\begin{aligned} e_{xx} &= l_1^2e_{\alpha\alpha} + m_1^2e_{\beta\beta} + n_1^2e_{\gamma\gamma} \qquad (xyz,\ 123) \\ e_{yz} &= 2(l_2l_3e_{\alpha\alpha} + m_2m_3e_{\beta\beta} + n_2n_3e_{\gamma\gamma}) \end{aligned} \tag{1.9}$$

*The symbol (λμν) denotes that other relationships are obtained by cyclic permutation of λμν. [Translator's note: Probably the (lmn) in (1.8) is intended.]

or

$$e_{\alpha\alpha} = l_1^2 e_{xx} + l_2^2 e_{yy} + l_3^2 e_{zz} + l_2 l_3 e_{yz} + l_1 l_3 e_{xz} + l_1 l_2 e_{xy} \quad (\alpha\beta\gamma,\ lmn)$$
$$e_{\beta\gamma} = 2(m_1 n_1 e_{xx} + m_2 n_2 e_{yy} + m_3 n_3 e_{zz}) + (m_2 n_3 + m_3 n_2)\, e_{yz} + + (m_1 n_3 + m_3 n_1)\, e_{xz} + (m_1 n_2 + m_2 n_1)\, e_{xy} = 0 \tag{1.10}$$

for the stress components

$$\sigma_x = l_1^2 \sigma_\alpha + m_1^2 \sigma_\beta + n_1^2 \sigma_\gamma \quad (xyz,\ 123)$$
$$\tau_{yz} = l_2 l_3 \sigma_\alpha + m_2 m_3 \sigma_\beta + n_2 n_3 \sigma_\gamma \tag{1.11}$$

or

$$\sigma_\alpha = l_1^2 \sigma_x + l_2^2 \sigma_y + l_3^2 \sigma_z + 2(l_2 l_3 \tau_{yz} + l_1 l_3 \tau_{xz} + l_1 l_2 \tau_{xy}) \quad (\alpha\beta\gamma,\ lmn)$$
$$\tau_{\beta\gamma} = m_1 n_1 \sigma_x + m_2 n_2 \sigma_y + m_3 n_3 \sigma_z + (m_2 n_3 + m_3 n_2)\, \tau_{yz} + + (m_1 n_3 + m_3 n_1)\, \tau_{xz} + (m_1 n_2 + m_2 n_1)\, \tau_{xy} = 0 \tag{1.12}$$

As we know [3], for stress components, and also strain components, there exist invariants which do not depend on the transformation of the coordinate axes. Along with the known invariants, we shall present one more relationship which contains components both of stress and of strain, and which also is invariant with respect to the transformation of the coordinate axis*.

$$I = \sigma_x e_{xx} + \sigma_y e_{yy} + \sigma_z e_{zz} + \tau_{xy} e_{xy} + \tau_{xz} e_{xz} + \tau_{yz} e_{yz} \tag{1.13}$$

We shall call this invariant a mixed invariant.

2. Proceeding from the existence of an elastic potential for the multimodulus body which we are considering, and also taking into account the linear relationships (1.7), the following can be obtained for the specific potential energy of strain [3]:

$$W = {}^1/_2 (\sigma_\alpha e_{\alpha\alpha} + \sigma_\beta e_{\beta\beta} + \sigma_\gamma e_{\gamma\gamma}) \tag{2.1}$$

Considering expression (2.1) and taking into account the mixed invariant (1.13), let us note that the specific potential energy of strain in the coordinate system x, y, z is expressed by the formula:

$$W = {}^1/_2 (\sigma_x e_{xx} + \sigma_y e_{yy} + \sigma_z e_{zz} + \tau_{xy} e_{xy} + \tau_{xz} e_{xz} + \tau_{yz} e_{yz}) \tag{2.2}$$

Thus, this formula, which is known in the theory of elasticity as the Clapeyron formula, also applies for the considered body. /66

*We shall not prove the invariancy of expression (1.13) in view of the elementary character and clumsiness of the derivations.

Substituting the values for the strains from (1.1) into (2.2), we obtain the following after certain transformations for W in stresses:

$$W = \tfrac{1}{2} a_{11} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) + a_{12} (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) + 2(a_{11} - a_{12}) (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) + \tfrac{1}{2} (a_{22} - a_{11}) \sigma_\beta^2 \tag{2.3}$$

Using the law of elasticity (1.4) and formula (1.6), the elastic potential (2.2) can be represented by the strain:

$$W = \frac{1}{2} (2A + B) (e_{xx}^2 + e_{yy}^2 + e_{zz}^2) + B (e_{xx}e_{yy} + e_{xx}e_{zz} + e_{yy}e_{zz}) + \frac{1}{2} A (e_{xy}^2 + e_{xz}^2 + e_{yz}^2) - \frac{1}{2} B_3 \frac{(2Ae_{\beta\beta} + B\theta)^2}{1 + B_3 (2A + B)} \tag{2.4}$$

3. It has been proven in the classical theory of elasticity that if the internal elastic forces have a potential, then, independently of the law of elasticity, the following Green formulas apply:

$$\sigma_x = \frac{\partial W}{\partial e_{xx}}, \quad \tau_{xy} = \frac{\partial W}{\partial e_{xy}} \quad (xyz) \tag{3.1}$$

It has also been proven that if an elastic body is governed by the generalized Hooke law, Castigliano formulas also apply for it:

$$e_{xx} = \frac{\partial W}{\partial \sigma_x}, \quad e_{xy} = \frac{\partial W}{\partial \tau_{xy}} \quad (xyz) \tag{3.2}$$

We shall now prove that Castigliano formulas also apply for the multimodulus material we are considering, which is characterized by the generalized law of elasticity (1.1) or (1.4) [2].

To do this, we shall calculate the partial derivative of W with respect to σ_x and shall indicate that it is equal to e_{xx}. On the basis of (2.3), we obtain:

$$\frac{\partial W}{\partial \sigma_x} = a_{11}\sigma_x + a_{12} (\sigma_y + \sigma_z) + \text{[illegible]} \frac{\partial \sigma_\beta}{\partial \sigma_x} \tag{3.3}$$

Comparing (3.3) with the first relationship (1.1), we note that in order to prove our statement it is necessary to indicate that

$$\partial\sigma_\beta / \partial\sigma_x = m_1^2 \tag{3.4}$$

Having calculated the partial derivative of σ_β (1.12) with respect to σ_x, we obtain

$$\frac{\partial\sigma_\beta}{\partial\sigma_x} = m_1^2 + 2\Big[m_1\frac{\partial m_1}{\partial\sigma_x}\sigma_x + m_2\frac{\partial m_2}{\partial\sigma_x}\sigma_y + m_3\frac{\partial m_3}{\partial\sigma_x}\sigma_z + \left(m_2\frac{\partial m_3}{\partial\sigma_x} + m_3\frac{\partial m_2}{\partial\sigma_x}\right)\tau_{yz} + \left(m_1\frac{\partial m_3}{\partial\sigma_x} + m_3\frac{\partial m_1}{\partial\sigma_x}\right)\tau_{xz} + \left(m_1\frac{\partial m_2}{\partial\sigma_x} + m_2\frac{\partial m_1}{\partial\sigma_x}\right)\tau_{xy}\Big] \qquad (3.5)$$

We shall now indicate that the expression enclosed in the brackets in formula (3.5) is equal to zero.

To do this, we differentiate the identity

$$m_1^2 + m_2^2 + m_3^2 = 1 \qquad (3.6)$$

with respect to σ_x and, on the basis of (1.8), we shall examine the following linear homogeneous system with respect to m_1, m_2, and m_3:

$$\begin{gathered} m_1\frac{\partial m_1}{\partial\sigma_x} + m_2\frac{\partial m_2}{\partial\sigma_x} + m_3\frac{\partial m_3}{\partial\sigma_x} = 0 \\ m_1n_1 + m_2n_2 + m_3n_3 = 0, \qquad m_1l_1 + m_2l_2 + m_3l_3 = 0 \end{gathered} \qquad (3.7)$$

It is evident from (3.6) that m_1, m_2, and m_3 cannot be equal to zero at the same time. Therefore, to satisfy system (3.7) it is sufficient that / 67

$$\begin{vmatrix} \partial m_1/\partial\sigma_x & \partial m_2/\partial\sigma_x & \partial m_3/\partial\sigma_x \\ n_1 & n_2 & n_3 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0 \qquad (3.8)$$

Since the last lines of the determinant (3.8), which are composed of the direction cosines of directions α and γ, cannot be proportional, the following is sufficient for the satisfaction of (3.8):

$$\text{either } \partial m_i/\partial\sigma_x = an_i, \text{ or } \partial m_i/\partial\sigma_x = bl_i \quad (i = 1,2,3) \qquad (3.9)$$

Here a and b are the proportionality factors. Let us note that in a general case a linear combination of the following form may apply:

$$\partial m_i/\partial\sigma_x = an_i + bl_i \; (i = 1, 2, 3)$$

However, the statements already presented essentially remain in force. Now let us note that when the first (second) condition of (3.9) is fulfilled, the expression enclosed in brackets in (3.5) becomes equal to zero due to formulas (1.12), where $\tau_{\beta\gamma} = 0$ ($\tau_{\alpha\beta} = 0$). Thus, the validity of formula (3.4) is proven.

Proceeding in a similar manner, the following group of formulas can be obtained:

$$\frac{\partial\sigma_\beta}{\partial\sigma_x} = m_1^2, \quad \frac{\partial\sigma_\beta}{\partial\sigma_y} = m_2^2, \quad \frac{\partial\sigma_\beta}{\partial\sigma_z} = m_3^2, \quad \frac{\partial\sigma_\beta}{\partial\tau_{xy}} = 2m_1m_2$$
$$\frac{\partial\sigma_\beta}{\partial\tau_{xz}} = 2m_1m_3, \quad \frac{\partial\sigma_\beta}{\partial\tau_{yz}} = 2m_2m_3 \tag{3.10}$$

The validity of Castigliano's formulas (3.2) becomes evident for the considered multimodulus material on the basis of these formulas.

4. On the basis of the Green (3.1) and the Castigliano (3.2) formulas, the Clapeyron formula (2.2) can be represented in two ways: e.g.,

$$\frac{\partial W}{\partial e_{xx}} e_{xx} + \frac{\partial W}{\partial e_{yy}} e_{yy} + \frac{\partial W}{\partial e_{zz}} e_{zz} + \frac{\partial W}{\partial e_{xy}} e_{xy} + \frac{\partial W}{\partial e_{xz}} e_{xz} + \frac{\partial W}{\partial e_{yz}} e_{yz} = 2W \tag{4.1}$$

$$\frac{\partial W}{\partial \sigma_x} \sigma_x + \frac{\partial W}{\partial \sigma_y} \sigma_y + \frac{\partial W}{\partial \sigma_z} \sigma_z + \frac{\partial W}{\partial \tau_{xy}} \tau_{xy} + \frac{\partial W}{\partial \tau_{xz}} \tau_{xz} + \frac{\partial W}{\partial \tau_{yz}} \tau_{yz} = 2W \tag{4.2}$$

The specific potential energy W can be considered in general as a function of only the stress components σ_x, ..., τ_{yz} or only of the strain components e_{xx}, ..., e_{yz}, since it is known that the principal stresses, (σ_α, σ_β, σ_γ), or the principal strains, ($e_{\alpha\alpha}$, $e_{\beta\beta}$, $e_{\gamma\gamma}$), and their direction cosines for the given stress or strain components, are determined by a single value. On the strength of this and on the basis of Euler's inverse theorem concerning heterogeneous functions [4], from (4.1) and (4.2) we can conclude that if we succeed in expressing W, (2.3) or (2.4) only by stresses σ_x, ..., τ_{yz} or only by strains e_{xx}, ..., e_{yz}, in the multimodulus theory of elasticity it will also be a homogeneous second-order function of its arguments.

5. Using the results presented above and proceeding in precisely the same manner as in the classical theory of elasticity [3], we can prove the validity of Clapeyron's theorem, and also the variation equations of Lagrange and Castigliano for the multimodulus materials which we are considering.

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